

# OBSERVATIONS CONCERNING GÖDEL'S 1931

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ABSTRACT. This article demonstrates the invalidity of Theorem VI in Gödel's monograph of 1931, by showing that

$$(15) \quad \overline{xB_{\kappa}(17 \text{ Gen } r)} \longrightarrow \text{Bew}_{\kappa}[Sb(r_{Z(x)}^{17})],$$

$$(16) \quad \overline{xB_{\kappa}(17 \text{ Gen } r)} \longrightarrow \text{Bew}_{\kappa}[\text{Neg}(Sb(r_{Z(x)}^{17}))],$$

(derived by definition (8.1)  $Q(x, y) \equiv \overline{xB_{\kappa}[Sb(y_{Z(y)}^{19})]}$  respectively from

$$(3) \quad R(x_1, \dots, x_n) \longrightarrow \text{Bew}_{\kappa}[Sb(r_{Z(x_1) \dots Z(x_n)}^{u_1 \dots u_n})],$$

$$(4) \quad \overline{R}(x_1, \dots, x_n) \longrightarrow \text{Bew}_{\kappa}[\text{Neg}(Sb(r_{Z(x_1) \dots Z(x_n)}^{u_1 \dots u_n}))],$$

of Theorem V) are false in  $P$ . This is achieved in two steps. First, the predicate complementary to the well-known Gödel's predicate  $\text{Bew}(x)$  is defined by adding a new relation  $\text{Wid}(x)$ . In accordance, new logical connections are established, Lemma (6). Second,

$$(I) \quad \overline{xB_k(17 \text{ Gen } r)} \rightarrow \text{Bew}_k[Sb(r_{Z(x)}^{17})],$$

$$(II) \quad \overline{xB_{\kappa}(17 \text{ Gen } r)} \rightarrow \text{Bew}_k[\text{Neg}(Sb(r_{Z(x)}^{17}))],$$

are derived from Lemma (6) and definition (8.1). It amounts to saying that (15) and (16) are false and unacceptable for the system  $(P)$ . On the account of that, the two well-known cases

1.  $17 \text{ Gen } r$  is not  $\kappa$ -PROVABLE,
2.  $\text{Neg}(17 \text{ Gen } r)$  is not  $\kappa$ -PROVABLE,

can not be drawn (unless we say that they are true because  $((15) \& (I)) \supset 1$ . and  $((16) \& (II)) \supset 2$ . are examples of  $(p \& \sim p) \supset q$ , i. e. *ex falso sequitur quodlibet*).

**Introduction.** “Wir müssen wissen. Wir werden wissen.” We must know. We will know. With these words, pronounced at the International Congress of Mathematicians in Paris, in 1900, Hilbert settled the basis of the logical investigations of the foundations of mathematics of the twentieth century. Whatever they were commands or wishes, they put the human spirit in a position of certainty about the solvability of every mathematical problem. These words embodied ancient aspirations, such as the achievement of a general problem-solving method, like Ramon Lull's *Ars Magna*, or the completion of a universal symbolic language like Leibnitz's *Characteristica* and Frege's *Begriffsschrift* [2, -4]. But for histories sense of humor, or for the unpredictability of human nature, they were just at the starting point of the exploration which led to the accomplishment of the opposite position, see for example the conviction of Godfrey Hardy in 1928.

“ There is of course no such theorem, and this is very fortunate, since if there were we should have a mechanical set of rules for the solution of all mathematical problems, and our activities as mathematicians would come to an end.” [15, -93]

It was hard to believe that statements about numbers like Fermat's Last Theorem and Goldbach's conjecture which the efforts of the centuries have failed to solve, could in fact be decided by a mechanical process. All this found its culmination in Gödel's 1931 article, that placed severe limits on the power of mathematical reasoning and on the power of the axiomatic method [10].

Many mathematicians at the end of the nineteenth-century considered consistency to be sufficient in securing the existence of theoretical objects. The result obtained by Gödel in 1930 assuring the existence of a model for the first order logic, apparently confirmed this opinion [9]. However, for Gödel comparing consistency with existence

“ manifestly presupposes the axiom that every mathematical problem is solvable. Or, more precisely, it presupposes that we cannot prove the unsolvability of any problem.” [8, 60-61]

Through Gödel's contributions, the reference to the existence of unsolvable problems in mathematics is recurring (see note 61 in [10, 190-191]), together with it's winding exchange with the existence of undecidable propositions [10, 144-145]. This connection of unsolvable problems and undecidable propositions, was supported by the conviction that

“ there are true propositions (which may even be provable by means of other principles) that cannot be derived in the system under consideration.” [9, 102-103]

So that, for Gödel, there were unsolvable problems in mathematics that, although contentually true, were unprovable in the formal system ( $P$  in [10, 150-151]).

“(Assuming the consistency of classical mathematics) one can even give examples of propositions (and in fact of those of the type of Goldbach or Fermat) that, while contentually true, are unprovable in the formal system of classical mathematics.” [11, 202-203]

Nowadays the developments concerning Fermat's last theorem [21, 20], clarify that *problems with no solution* are not so *unprovable* in mathematics. We are then allowed to state that, in Gödel's convictions, around the existence of undecidable problems in mathematics, there is a presupposition of unprovability that the present article will help to clarify.

Of all the remarkable logical achievements of the twentieth century perhaps the most outstanding is Gödel's celebrated incompleteness argumentation of 1931. In contrast to Hilbert's program which called for embodying classical mathematics in a formal system and proving that system consistent by finitary methods [13, 14], Gödel's paper showed that not even the first step could be carried out fully, any formal system suitable for the arithmetic of integers was incomplete [10, 12].

Gödel's incompleteness argument holds today the same scientific status as Einstein's principle of relativity, Heisenberg's uncertainty principle, and Watson and Crick's double helix model of DNA [3, -315]. With absolute respect and acknowledgement for the extraordinary contribution given by Gödel to the logical investigation, this article brings Gödel's achievement into question by the definition of the refutability predicate as a number-theoretical statement.

As is well known, Gödel numbering is a special device by which each expression in arithmetic can refer to itself. Just a declarative programmer can catch the extraordinary expressive power of Gödel numbering.

“Many of the logicians who worked in the first half of this century were actually the first inventors of programming languages. Gödel uses a scheme for expressing algorithms that is much like the high-level language LISP ... .” [7, -19] “But if you read his original 1931 paper with the benefit of hindsight, you’ll see that Gödel is programming in LISP, ... .” [7, -14]

It was precisely with this view of Gödel programming recursive statements that the content of this work was conceived. According to Gödel’s first incompleteness argument it is possible to construct a formally undecidable proposition in PM, a statement that, although true, turns out to be neither provable nor refutable for the system. This article develops proof of invalidity of Theorem VI in Gödel’s 1931, the so-called Gödel’s first incompleteness theorem, in two steps: defining refutability within the same recursive status as provability and showing that as a consequence propositions (15) and (16), derived from definition 8.1 in Gödel’s 1931 [10, 174-175], are false and unacceptable for the system. The achievement of their falsity blocks the derivation of Theorem VI, which turns out to be therefore invalid, together with all the depending theorems (Sections 1, 2 and 3).

Completeness is connected to decidability since a resolution procedure takes advantages of those sorts of logical inferences formalized also by the axiomatic systems. With respect to the finitely axiomatized systems, it leads to a theoretical list of all theorems, which is in turn a procedure of semi-decidability for the first order logic. With regard to that, the Lemmas of Section 1 are clarifying and opening new perspectives, partially already exposed in [4, 6]. In fact this work applies directly to the text of Gödel’s 1931 article, the results obtained in [4] for the modern version of the incompleteness argument, which is mainly based on the diagonalization lemma. It provides therefore as a novel contribution clear and advanced proof of the invalidity of Gödel’s original argument. For all these aspects the result exposed here stands out from previous criticism of Gödel’s 1931, mainly focused on the antinomic features of Gödel’s self-reference statement, (see [16], [1], [18], resumed in [17]; [19]). In the following the reader is required to be familiar with Gödel’s 1931 [10].

# 1.

Let us begin adding to the list of functions (relations) 1-45 in Gödel’s 1931 two new relations, 45.1 and 46.1, in terms of the preceding ones by the procedures given in Theorems I-IV [10, 158-163]. We shall recall only the well-known definitions 44., 45. and 46., for the whole list the reader is referred to [10, 162-171].

$$44. Bw(x) \equiv (n)\{0 < n \leq l(x) \longrightarrow Ax(n Gl x) \vee (Ep, q)[0 < p, q < n \& Fl(n Gl x, p Gl x, q Gl x)]\} \& l(x) > 0,$$

$x$  is a PROOF ARRAY (a finite sequence of FORMULAS, each of which is either an AXIOM or an IMMEDIATE CONSEQUENCE of two of the preceding FORMULAS).

$$45. xBy \equiv Bw(x) \& [l(x)] Gl x = y,$$

$x$  is a PROOF of the FORMULA  $y$ .

$$45.1. xWy \equiv Bw(x) \& [l(x)] Gl x = \text{Neg}(y),$$

$x$  is a REFUTATION of the FORMULA  $y$ .

46.  $\text{Bew}(x) \equiv (Ey)yBx$ ,  
 $x$  is a PROVABLE FORMULA. ( $\text{Bew}(x)$  is the first one of the notions 1–46 of which we cannot assert that it is recursive.)

46.1.  $\text{Wid}(x) \equiv (Ez)zWx$ ,  
 $x$  is a REFUTABLE FORMULA. ( $\text{Wid}(x)$  is the second one of the notions 1–46 of which we cannot assert that it is recursive.)  $\text{Wid}$  is the shortening for “Widerlegung” and must not be mistaken with the notion defined by Gödel in note 63 referring instead to “Widerspruchsfrei” [10, 192-193], which afterwards we will call  $\text{Wid}_s$ .

Being classes included among relations, as one-place relation, 44. define the recursive class of the proof arrays. Recursive relations  $R$  have the property that for every given  $n$ -tuple of numbers it can be decided whether  $R(x_1 \dots x_n)$  holds or not. 45. and 45.1 define recursive relations,  $xB y$  and  $xW y$ , so that for every given couple of numbers it can be decided whether  $xB y$  and  $xW y$  hold or not. In accordance we can state the following Lemmas.

**Lemma 1.**

$$(x)\{\text{Wid}(x) \sim \text{Bew}[\text{Neg}(x)]\} \text{ and } (x)\{\text{Bew}(x) \sim \text{Wid}[\text{Neg}(x)]\}.$$

*Proof.* Immediately by 46.1., 46., 45.1., 45. and  $\text{Neg}(\text{Neg}(x)) \sim x$ .  $\square$

**Lemma 2.** For any  $x, y$  in  $P$ ,

$$\text{not both } xW y \text{ and } xB y.$$

*Proof.* Let us suppose to have, for an arbitrary couple  $x, y$ , both  $xW y$  and  $xB y$  in  $P$ . By 45. and 45.1

$$Bw(x) \ \& \ [l(x)] \ Gl \ x = y \quad \& \quad Bw(x) \ \& \ [l(x)] \ Gl \ x = \text{Neg}(y),$$

then by 44.

$$\begin{aligned} (n)\{0 < n \leq l(x) \longrightarrow Ax(n \ Gl \ x) \vee \\ (Ep, q)[0 < p, q < n \ \& \ Fl(n \ Gl \ x, p \ Gl \ x, q \ Gl \ x)]\} \ \& \\ l(x) > 0 \ \& \ [l(x)] \ Gl \ x = y \ \& \ [l(x)] \ Gl \ x = \text{Neg}(y). \end{aligned}$$

For  $l(x) > 0$  and  $n = l(x)$  this should be

$$\begin{aligned} Ax(y) \ \& \ Ax(\text{Neg}(y)) \vee (Ep, q)[0 < p, q < n \ \& \\ Fl(y, p \ Gl \ x, q \ Gl \ x) \ \& \ Fl(\text{Neg}(y), p \ Gl \ x, q \ Gl \ x)] \end{aligned}$$

which is impossible by 42.  $Ax(x)$ , 43.  $Fl(x, y, z)$  and by the definitions of the axioms and of immediate consequence in  $P$ .  $\square$

Recursive functions have the property that, for each given set of values of the arguments, the value of the function can be computed by a finite procedure [12, -348, 369-371]. The following Lemma, especially as regard to  $l$ ,  $Gl$  and  $\text{Neg}$ , is involved with this property and the today’s notion of effective computability.

**Lemma 3.** For any  $x, y$  such that  $x$  is a PROOF of the FORMULA  $y$ , in  $P$ ,

$$xB y \text{ or } xW y.$$

*Proof.* Assume that  $x$  is a PROOF of the FORMULA  $y$ . Then, 45.,  $xB y$ , and from the schema  $p \supset p \vee q$ , II.2. [10, 154-155],  $xB y \supset xB y \vee xW y$  so that  $xB y \vee xW y$ , i. e. for any  $x, y$  such that  $x$  is a PROOF of the FORMULA  $y$ ,  $xB y \vee xW y$ .  $\square$

In the case that  $x$  is a PROOF of the FORMULA  $\text{Neg}(y)$ , i.e.  $[l(x)] Gl x = \text{Neg}(y)$ , the same Lemma can be derived as follows. Given  $Bw(x)$ ,  $[l(x)] Gl x$  is a natural number for a FORMULA  $z$ .  $\text{Neg}$  is a recursive function so that we have a finite procedure to determine the value of  $\text{Neg}(y)$ . Let us suppose that  $z = \text{Neg}(y)$ , then  $xB z$  is true, and from  $xB z \supset xB z \vee xW z$  we draw  $xB z \vee xW z$ . We notice that this Lemma holds for any FORMULA  $z$  such that  $[l(x)] Gl x = z$  even if  $xB z$  is  $xW y$ .

**Lemma 4.** For any  $x, y$  such that  $x$  is a PROOF of the FORMULA  $y$ , in  $P$ ,

$$xW y \sim \overline{xB y}.$$

*Proof.* Immediately by Lemma (2) and Lemma (3).  $\square$

Let us notice that Lemma (4) yields, for any  $x, y$  such that  $x$  is a PROOF of the FORMULA  $y$  in  $P$ , also  $xW y \sim xB y$ .

**Lemma 5.** For any  $x, y$  such that  $x$  is a PROOF of the FORMULA  $y$ , in  $P$ ,

$$\overline{\text{Bew}(y)} \text{ if } \overline{xB y} \text{ and } \overline{\text{Wid}(y)} \text{ iff } xB y.$$

*Proof.* Let us assume that  $x$  is a PROOF of the FORMULA  $y$  and  $\overline{xB y}$ . As an immediate consequence we have  $(x)\overline{xB y}$ . By 46.  $\text{Bew}(y) \sim (Ex)xB y$ ,  $\overline{\text{Bew}(y)} \sim \overline{(Ex)xB y}$  and  $\overline{\text{Bew}(y)} \sim (x)\overline{xB y}$ . Accordingly for any  $x, y$  such that  $x$  is a PROOF of the FORMULA  $y$ ,  $\overline{\text{Bew}(y)}$  if  $\overline{xB y}$ .

Let us assume that  $x$  is a PROOF of the FORMULA  $y$ . By 45.  $xB y$  and from Lemma (4)  $\overline{xW y} \sim xB y$ , so that  $\overline{xW y}$ .  $(x)\overline{xW y}$  is an immediate consequence of  $\overline{xW y}$ , hence  $(x)\overline{xW y}$ .  $(p \& q) \supset (p \sim q)$  yields then  $(x)\overline{xW y} \sim xB y$ . By 46.1  $\text{Wid}(y) \sim (Ex)xW y$ , so that  $\overline{\text{Wid}(y)} \sim \overline{(Ex)xW y}$  and  $\overline{\text{Wid}(y)} \sim (x)\overline{xW y}$ . Accordingly, for any  $x, y$  such that  $x$  is a PROOF of the FORMULA  $y$ ,  $\overline{\text{Wid}(y)} \sim xB y$ .  $\square$

All preceding Lemmas were carried out constructively, needlessly to assume consistency. Let us recall the following Gödel's definitions [10, -173]. Let  $\kappa$  be any class of FORMULAS and  $Flg(\kappa)$  the smallest set of FORMULAS that contains all FORMULAS of  $\kappa$  and all AXIOMS, and is closed under the relation IMMEDIATE CONSEQUENCE.

$$(5) \quad \begin{aligned} Bw_\kappa(x) \equiv & (n)[n \leq l(x) \longrightarrow Ax(n Gl x) \vee (n Gl x) \in \kappa \vee \\ & (Ep, q)\{0 < p, q < n \& Fl(n Gl x, p Gl x, q Gl x)\}] \& l(x) > 0, \end{aligned}$$

$$(6) \quad xB_\kappa y \equiv Bw_\kappa(x) \& [l(x)] Gl x = y,$$

$$(6.1) \quad \text{Bew}_\kappa(x) \equiv (Ey)yB_\kappa x,$$

$$(7) \quad (x)[\text{Bew}_\kappa(x) \sim x \in Flg(\kappa)].$$

Let us augment this list with two new definitions.

$$(6.2) \quad xW_\kappa y \equiv Bw_\kappa(x) \& [l(x)] Gl x = \text{Neg}(y),$$

$$(6.2.1) \quad \text{Wid}_\kappa(x) \equiv (Ey)yW_\kappa x.$$

We add further that (6.1) and (7) yield

$$(7.1) \quad (x)[(Ey)yB_\kappa x \sim x \in \text{Flg}(\kappa)].$$

**Lemma 6.** *For any  $x$  such that  $Bw_\kappa(x)$  and any FORMULA  $y$  such that  $[l(x)] Gl x = y$  in  $\kappa$*

$$\overline{Bew_\kappa(y)} \text{ if } \overline{x B_\kappa y} \text{ and } \overline{\text{Wid}_\kappa(y)} \text{ iff } x B_\kappa y.$$

*Proof.* By Lemma (5) and previous definitions.  $\square$

**Lemma 7.** *Given a CLASS SIGN  $a$  with the FREE VARIABLE  $v$  and a SIGN  $c$  of the same type as  $v$ ,*

- (i) *if  $\overline{Bew_\kappa(v \text{ Gen } a)}$  then  $\overline{Bew_\kappa[Sb(a_c^v)]}$ ,*
- (ii) *if  $\overline{\text{Wid}_\kappa(v \text{ Gen } a)}$  then  $\overline{\text{Wid}_\kappa[Sb(a_c^v)]}$ .*

*Proof.* (i) Let us suppose that  $\overline{Bew_\kappa[Sb(a_c^v)]}$ , then from (7)  $Sb(a_c^v) \in \text{Flg}(\kappa)$ .  $\text{Flg}(\kappa)$  is closed under the relation IMMEDIATE CONSEQUENCE therefore  $v \text{ Gen } a \in \text{Flg}(\kappa)$ . Thus, by (7) again,  $Bew_\kappa(v \text{ Gen } a)$ . Accordingly if  $\overline{Bew_\kappa(v \text{ Gen } a)}$  then  $\overline{Bew_\kappa[Sb(a_c^v)]}$ .

(ii) Let us suppose that  $\overline{\text{Wid}_\kappa(v \text{ Gen } a)}$ , then, by Lemma (6), for any  $x$  such that  $Bw_\kappa(x)$  and  $[l(x)] Gl x = v \text{ Gen } a$  in  $\kappa$ ,  $x B_\kappa(v \text{ Gen } a)$ . By (7.1)<sup>1</sup> we have therefore  $v \text{ Gen } a \in \text{Flg}(\kappa)$ . Thanks to Axiom III.1<sup>2</sup>, we obtain  $Sb(a_c^v) \in \text{Flg}(\kappa)$  and from (7.1) again, we have  $(Ez)z B_\kappa[Sb(a_c^v)]$ . Finally, from Lemma (6) we have  $\overline{\text{Wid}_\kappa[Sb(a_c^v)]}$  (for any  $x$  such that  $Bw_\kappa(x)$  and  $[l(x)] Gl x = v \text{ Gen } a$ ).  $\square$

2.

We are now ready to derive the main result of this article, Theorem (8). The invalidity of Theorem VI in Gödel's 1931 article [10, 172-177] follows from Lemmas (6) and (7). In Gödel's 1931 argumentation the proof that both  $17 \text{ Gen } r$  and  $\text{Neg}(17 \text{ Gen } r)$  are not  $\kappa$ -PROVABLE is based on the two statements [10, 174-176]

$$(15) \quad \overline{x B_\kappa(17 \text{ Gen } r)} \longrightarrow \text{Bew}_\kappa[Sb(r_{Z(x)}^{17})]$$

$$(16) \quad x B_\kappa(17 \text{ Gen } r) \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(x)}^{17}))],$$

which are respectively deduced from  $Q(x, y)$  and  $\overline{Q}(x, y)$ , whereas

$$(8.1) \quad Q(x, y) \equiv \overline{x B_\kappa[Sb(y_{Z(y)}^{19})]}.$$

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<sup>1</sup>If, for any  $x$  such that  $Bw_\kappa(x)$  and  $[l(x)] Gl x = v \text{ Gen } a$ ,  $x B_\kappa(v \text{ Gen } a)$  then, for any  $x$  such that  $Bw_\kappa(x)$  and  $[l(x)] Gl x = v \text{ Gen } a$ ,  $(Ey)y B_\kappa(v \text{ Gen } a)$ .

<sup>2</sup>III. Any formula that results from the schema

$$1. v \Pi(a) \supset \text{Subst } a(c^v)$$

when the following substitutions are made for  $a$ ,  $v$ , and  $c$  (and the operation indicated by Subs is performed in 1): for  $a$  any formula, for  $v$  any variable and for  $c$  any sign of the same type as  $v$ , provided  $c$  does not contain any variable that is bound in  $a$  at a place where  $v$  is free [10, 154-155; 37. 38. 168-169].

More precisely,  $Q(x, y)$  is an instance of  $R(x_1, \dots, x_n)$  in

$$(3) \quad R(x_1, \dots, x_n) \longrightarrow \text{Bew}_\kappa[\text{Sb}(r_{Z(x_1)\dots Z(x_n)}^{u_1\dots u_n})],$$

as exactly as  $\overline{Q}(x, y)$  is an example of  $\overline{R}(x_1, \dots, x_n)$  in

$$(4) \quad \overline{R}(x_1, \dots, x_n) \longrightarrow \text{Bew}_\kappa[\text{Neg}(\text{Sb}(r_{Z(x_1)\dots Z(x_n)}^{u_1\dots u_n}))],$$

(Theorem V [10, 170-171]), so that (3)  $\longrightarrow$  (9)  $\longrightarrow$  (15) and (4)  $\longrightarrow$  (10)  $\longrightarrow$  (16) [10, 170-175].

In accordance with Lemmas (6) and (7) we will show that given  $Q(x, y)$ , (15) turns out to be false, and, similarly, given  $\overline{Q}(x, y)$ , (16) results to be false. We will show then as a consequence that Theorem VI in Gödel's 1931 [10, 172-173] is not achievable.

*Proof.* Let us assume  $Q(x, y)$  in  $\kappa$ , then by definition (8.1)

$$(I.1) \quad \overline{xB_\kappa[\text{Sb}(y_{Z(y)}^{19})]}.$$

We substitute in it  $p$  for  $y$ , see definitions

$$(11) \quad p = 17 \text{ Gen } q,$$

( $p$  is a CLASS SIGN with the FREE VARIABLE 19),

$$(12) \quad r = \text{Sb}(q_{Z(p)}^{19})$$

( $r$  is a recursive CLASS SIGN with the FREE VARIABLE 17) and

$$(13) \quad \text{Sb}(p_{Z(p)}^{19}) = \text{Sb}([17 \text{ Gen } q]_{Z(p)}^{19}) = 17 \text{ Gen } \text{Sb}(q_{Z(p)}^{19}) = 17 \text{ Gen } r$$

in [10, 174-175], so that we have

$$(I.2) \quad \overline{xB_\kappa(17 \text{ Gen } r)}.$$

By  $a \rightarrow (\neg b \rightarrow \neg(a \rightarrow b))$  we have

$$(I.3) \quad \overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \left[ \overline{\text{Bew}_\kappa[\text{Sb}(r_{Z(x)}^{17})]} \longrightarrow \overline{\overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \text{Bew}_\kappa[\text{Sb}(r_{Z(x)}^{17})]} \right],$$

hence, by (I.2),

$$(I.4) \quad \overline{\text{Bew}_\kappa[\text{Sb}(r_{Z(x)}^{17})]} \longrightarrow \overline{\overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \text{Bew}_\kappa[\text{Sb}(r_{Z(x)}^{17})]}.$$

Lemma (6) yields for any  $x$  such that  $Bw_\kappa(x)$ , and  $[l(x)] Gl x = 17 \text{ Gen } r$ ,

$$\overline{\text{Bew}_\kappa(17 \text{ Gen } r)} \text{ if } \overline{xB_\kappa(17 \text{ Gen } r)},$$

accordingly from (I.2)

$$(I.5) \quad \overline{\text{Bew}_\kappa(17 \text{ Gen } r)},$$

and by Lemma (7) (i)

$$(I.6) \quad \overline{\text{Bew}_\kappa[\text{Sb}(r_{Z(x)}^{17})]}.$$

(I.4) and (I.6) yield, for any  $x$  such that  $Bw_\kappa(x)$ , and  $[l(x)] Gl x = 17 \text{ Gen } r$ ,

$$(I) \quad \overline{\overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \text{Bew}_\kappa[\text{Sb}(r_{Z(x)}^{17})]}.$$

□

*Proof.* Let us assume  $\overline{Q}(x, y)$  in  $\kappa$ , so that by (8.1), substituting  $p$  for  $y$ , (11), (12) and (13), we obtain

$$(II.1) \quad xB_\kappa(17 \text{ Gen } r).$$

From  $a \rightarrow (\sim b \rightarrow \sim(a \rightarrow b))$ ,

$$(II.2) \quad xB_\kappa(17 \text{ Gen } r) \longrightarrow \left[ \overline{\text{Wid}_\kappa[Sb(r_{Z(x)}^{17})]} \longrightarrow \overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \overline{\text{Wid}_\kappa[Sb(r_{Z(x)}^{17})]} \right],$$

and then

$$(II.3) \quad \overline{\text{Wid}_\kappa[Sb(r_{Z(x)}^{17})]} \longrightarrow \overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \overline{\text{Wid}_\kappa[Sb(r_{Z(x)}^{17})]}.$$

From Lemma (6), for any  $x$  such that  $Bw_\kappa(x)$ , and  $[l(x)] Gl x = 17 \text{ Gen } r$

$$\overline{\text{Wid}_\kappa(17 \text{ Gen } r)} \text{ iff } xB_\kappa(17 \text{ Gen } r),$$

and II.1

$$(II.4) \quad \overline{\text{Wid}_\kappa(17 \text{ Gen } r)}.$$

Lemma (7) (ii)<sup>3</sup> yields, for any  $x$  such that  $Bw_\kappa(x)$ , and  $[l(x)] Gl x = 17 \text{ Gen } r$ ,

$$(II.5) \quad \overline{\text{Wid}_\kappa[Sb(r_{Z(x)}^{17})]}$$

and from II.3

$$(II.6) \quad \overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \overline{\text{Wid}_\kappa[Sb(r_{Z(x)}^{17})]}.$$

Finally, by Lemma (1), for any  $x$  such that  $Bw_\kappa(x)$ , and  $[l(x)] Gl x = 17 \text{ Gen } r$ ,

$$(II) \quad \overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \overline{\text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(x)}^{17}))]}.$$

□

By (I) and (II), (15) and (16) turn out to be false for any  $x$  such that  $x$  is a *PROOF ARRAY* which last FORMULA is  $17 \text{ Gen } r$  in  $\kappa$ , and the demonstration of Theorem VI cannot be accomplished. Indeed, for  $n$  such that  $Bw_\kappa(n)$  and  $[l(n)] Gl n = 17 \text{ Gen } r$ , (I) and (II) yield in  $\kappa$

$$\overline{\overline{nB_\kappa(17 \text{ Gen } r)}} \longrightarrow \overline{\text{Bew}_\kappa[Sb(r_{Z(n)}^{17})]}$$

and

$$\overline{nB_\kappa(17 \text{ Gen } r)} \longrightarrow \overline{\text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(n)}^{17}))]},$$

hence, within the case

“1.  $17 \text{ Gen } r$  is not  $\kappa$ -PROVABLE”

in Gödel's 1931 [10, 176-177],  $\text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(n)}^{17}))]$  has no basis to be obtained from  $nB_\kappa(17 \text{ Gen } r)$  in  $\kappa$ , therefore no proof that  $17 \text{ Gen } r$  is not  $\kappa$ -PROVABLE can be achieved.

Accordingly,  $(n) \overline{nB_\kappa(17 \text{ Gen } r)}$  has no soundness as a consequence of (6.1) within the next case

---

<sup>3</sup> $Z(x)$  does not contain any variable bound in  $r$  at a place at which 17 is free. This mainly because  $Z(x)$  is by definition, 17.  $Z(n)$  [10, 164-165], a NUMERAL and as such does not contain any variable. But if we think to  $x$  in  $Z(x)$  as to the *PROOF ARRAY* such that  $Bw_\kappa(x)$  and  $[l(x)] Gl x = 17 \text{ Gen } r$ ,  $x$  cannot be a variable bound in  $r$  at a place at which 17 is free, since  $x > r$  (see definitions 6.  $nGl x$  and 7.  $l(x)$  [10, 162-163]) and, as it can be shown by induction on the recursive definition 44.  $Bw(x)$ ,  $x$  is not in  $r$ .



“2.  $\text{Neg}(17 \text{ Gen } r)$  is not  $\kappa$ -PROVABLE” [10, 176-177]. Moreover, for  $n$  such that  $Bw_\kappa(n)$  and  $[l(n)] Gl n = 17 \text{ Gen } r$ ,  $\text{Bew}_\kappa[Sb(r_{Z(n)}^{17})]$  is not a consequence of  $nB_\kappa(17 \text{ Gen } r)$  in  $\kappa$ , so that neither a demonstration that  $\text{Neg}(17 \text{ Gen } r)$  is not  $\kappa$ -PROVABLE can be accomplished.

Consequently the statement of Theorem VI, “For every  $\omega$ -consistent recursive class  $\kappa$  of FORMULAS there are recursive CLASS SIGNS  $r$  such that neither  $v \text{ Gen } r$  nor  $\text{Neg}(v \text{ Gen } r)$  belongs to  $\text{Flg}(\kappa)$  (where  $v$  is the FREE VARIABLE of  $r$ )” [10, 172-173], has no proof.

Furthermore, the assertion that “it suffices for the existence of propositions undecidable that the class  $\kappa$  be  $\omega$ -consistent”, is now meaningless [10, 176-177]. We can then state the following theorem.

**Theorem 8.** *The existence of undecidable propositions of the form  $v \text{ Gen } r$  is not a theorem in  $\kappa$ .*

The invalidity of theorems VIII, IX and XI, all consequent of theorem VI, follows immediately [10, 184-194]. In the outlined derivation of theorem XI [10, 192-194], the assertion that  $17 \text{ Gen } r$  is not  $\kappa$ -PROVABLE, together with the assumption about the consistency of  $\kappa$ , have now no justification. Therefore the statements  $\text{Wid}_s(\kappa) \longrightarrow \text{Bew}_\kappa(17 \text{ Gen } r)$  (23) and  $\text{Wid}_s(\kappa) \longrightarrow (x)Q(x, p)$  (24) are in  $P$  without soundness ( $\text{Wid}_s(\kappa)$  means “ $\kappa$  is consistent” and is defined by  $\text{Wid}_s(\kappa) \equiv (Ex)(\text{Form}(x) \& \overline{\text{Bew}_\kappa}(x))$ , see note 63 [10, 192-193]).

### 3.

To resume briefly, in Gödel’s 1931, (15) and (16) are derived by means of definition (8.1)  $Q(x, y)$  from (3) and (4) taking respectively  $Q(x, y)$  as an instance of  $R(x_1, \dots, x_n)$  and  $\overline{Q}(x, y)$  as an instance of  $\overline{R}(x_1, \dots, x_n)$ , and then the two cases “1.  $17 \text{ Gen } r$  is not  $\kappa$ -PROVABLE” and “2.  $\text{Neg}(17 \text{ Gen } r)$  is not  $\kappa$ -PROVABLE” are derived respectively from (15) and (16) (figure 1).

Any deduction is sound at whatever row if it is sound at all the preceding rows. Within the whole of Gödel’s 1931 the propositions (15) and (16) are meant to be sound to secure the correctness of the conclusions. As showed in Sections 1 and 2 this is not the case. By Lemmas (6) and (7), given  $Q(x, y)$ , the negation of (15), (I), can be deduced and, given  $\overline{Q}(x, y)$ , we can obtain the negation of (16), (II). This invalidates the deduction of propositions (15) and (16) (figure 2). Indeed the soundness of “1.  $17 \text{ Gen } r$  is not  $\kappa$ -PROVABLE” and “2.  $\text{Neg}(17 \text{ Gen } r)$  is not  $\kappa$ -PROVABLE” is based respectively on the soundness of (15) and (16). As we showed (15) and (16) are false (figure 3), therefore there are no sound bases in Gödel’s 1931 to state Theorem VI, Theorem (8).

A definition of a new relation, like 8.1, is supposed to be true at the row it is asserted. Let us consider this definition at the light of our results. For  $Q(x, y)$  as a true relation we have both

$$\begin{aligned} Q(x, y) &\longrightarrow \overline{x B_\kappa(17 \text{ Gen } r)} \longrightarrow \text{Bew}_\kappa[Sb(r_{Z(x)}^{17})], \\ Q(x, y) &\longrightarrow \overline{\overline{x B_\kappa(17 \text{ Gen } r)}} \longrightarrow \overline{\text{Bew}_\kappa[Sb(r_{Z(x)}^{17})]}, \end{aligned}$$

i. e. definition (8.1) implies a formula and its negation. A ancient tradition identifies a contradiction with a couple of propositions where one is the negation

of the other. If  $Q(x, y)$  is considered to be true then a contradiction follows, the couple (15) and (I). Such a couple can be regarded from two distinguished point of views: the consequences for  $Q(x, y)$  and the consequences for  $P$  (and  $\kappa$ ). The first point-view delivers directly to the antinomic features of  $Q(x, y)$ . The couple (15) and (I) is always false and can be derived when  $Q(x, y)$  is supposed to be true, i. e.  $Q(x, y) \rightarrow \text{false}$ . As a first immediate consequence,  $Q(x, y)$  is false too. But as showed above if  $\overline{Q}(x, y)$  then we have both (16) and (II), a contradictory couple again. All that clarify the antinomic features of 8.1 and leads to re-consider it in terms of the theory of definition along the lines already developed in [5]. From the second point-view of the consequences for  $P$ , if  $Q(x, y) \rightarrow \text{false}$  then *ex falso sequitur quodlibet*, ( $\sim a \rightarrow (a \rightarrow b)$ ). The existence of the undecidable formula in Gödel's 1931 follows *ex falso*, i. e. it follows because everything follows from false. The law ( $\sim a \rightarrow (a \rightarrow b)$ ) is intuitionistically accepted but it leads  $P$  to collapse.  $P$  is inconsistent, in it everything follows, even the existence of a property for which *it is possible neither to give a counterexample nor to prove that it holds of all numbers*.

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- (1) ...
- (2) ...
- (3)  $R(x_1, \dots, x_n) \longrightarrow \text{Bew}_\kappa[Sb(r_{Z(x_1)\dots Z(x_n)}^{u_1\dots u_n})]$
- (4)  $\overline{R}(x_1, \dots, x_n) \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(x_1)\dots Z(x_n)}^{u_1\dots u_n}))]$
- (5) ...
- (6) ...
- (6.1) ...
- (7) ...
- (8) ...
- (8.1)  $Q(x, y)$   
 $Q(x, y) \longrightarrow \left[ \overline{xB_\kappa[Sb(y_{Z(y)}^{19})]} \longrightarrow \text{Bew}_\kappa[Sb(q_{Z(x)Z(y)}^{17}{}^{19})] \right]$   
 $\overline{Q}(x, y) \longrightarrow \left[ xB_\kappa[Sb(y_{Z(y)}^{19})] \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(q_{Z(x)Z(y)}^{17}{}^{19}))] \right]$
- (9)  $xB_\kappa[Sb(y_{Z(y)}^{19})] \longrightarrow \text{Bew}_\kappa[Sb(q_{Z(x)Z(y)}^{17}{}^{19})]$
- (10)  $xB_\kappa[Sb(y_{Z(y)}^{19})] \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(q_{Z(x)Z(y)}^{17}{}^{19}))]$
- (11) ...
- (12) ...  
(11), (12)  $\longrightarrow$  (13)  
(12)  $\longrightarrow$  (14)
- (13) ...
- (14) ...  
(13), (14), (9)  $\longrightarrow \left[ \overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \text{Bew}_\kappa[Sb(r_{Z(x)}^{17})] \right]$   
(13), (14), (10)  $\longrightarrow \left[ xB_\kappa(17 \text{ Gen } r) \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(x)}^{17}))] \right]$
- (15)  $\overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \text{Bew}_\kappa[Sb(r_{Z(x)}^{17})]$
- (16)  $xB_\kappa(17 \text{ Gen } r) \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(x)}^{17}))]$   
(15)  $\longrightarrow$  "1. 17 Gen r is not  $\kappa$ -PROVABLE"  
(16)  $\longrightarrow$  "2. Neg(17 Gen r) is not  $\kappa$ -PROVABLE"

---

"1. 17 Gen r is not  $\kappa$ -PROVABLE"  
"2. Neg(17 Gen r) is not  $\kappa$ -PROVABLE"

FIGURE 1. Gödel's 1931

$$\begin{array}{ll}
(1) & \dots \\
(2) & \dots \\
(3) & \dots \\
(4) & \dots \\
(5) & \dots \\
(6) & \dots \\
(6.1) & \dots \\
(7) & \dots \\
(8) & \dots \\
(8.1) & Q(x, y) \\
& Q(x, y) \longrightarrow \left[ \overline{xB_\kappa[Sb(y_{Z(y)}^{19})]} \longrightarrow \text{Bew}_\kappa[Sb(q_{Z(x)Z(y)}^{17}{}^{19})] \right] \\
& \overline{Q}(x, y) \longrightarrow \left[ xB_\kappa[Sb(y_{Z(y)}^{19})] \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(q_{Z(x)Z(y)}^{17}{}^{19}))] \right] \\
(9) & \overline{xB_\kappa[Sb(y_{Z(y)}^{19})]} \longrightarrow \text{Bew}_\kappa[Sb(q_{Z(x)Z(y)}^{17}{}^{19})] \\
(10) & xB_\kappa[Sb(y_{Z(y)}^{19})] \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(q_{Z(x)Z(y)}^{17}{}^{19}))] \\
(11) & \dots \\
(12) & \dots \\
& (11), (12) \longrightarrow (13) \\
& (12) \longrightarrow (14) \\
(13) & \dots \\
(14) & \dots \\
& (13), (14), (9) \longrightarrow \left[ \overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \text{Bew}_\kappa[Sb(r_{Z(x)}^{17})] \right] \\
& (13), (14), (10) \longrightarrow \left[ xB_\kappa(17 \text{ Gen } r) \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(x)}^{17}))] \right]
\end{array}$$


---


$$\begin{array}{ll}
(15) & \overline{xB_\kappa(17 \text{ Gen } r)} \longrightarrow \text{Bew}_\kappa[Sb(r_{Z(x)}^{17})] \\
(16) & xB_\kappa(17 \text{ Gen } r) \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(x)}^{17}))]
\end{array}$$

FIGURE 2. Derivation of (15) and (16) within Gödel's 1931

(1)	...
(2)	...
(3)	...
(4)	...
(5)	...
(6)	...
(6.1)	...
(6.2)	...
(6.2.1)	...
(7)	...
(7.1)	...
	Lemma (6)
	Lemma (7)
(8)	...
(8.1)	$Q(x, y)$
(11)	...
(12)	...
(13)	...
	$Q(x, y) \longrightarrow \overline{\overline{xB_\kappa(17 \text{ Gen } r)}} \longrightarrow \text{Bew}_\kappa[Sb(r_{Z(x)}^{17})]$
	$\overline{Q}(x, y) \longrightarrow \overline{\overline{xB_\kappa(17 \text{ Gen } r)}} \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(x)}^{17}))]$
(I)	$\overline{\overline{xB_\kappa(17 \text{ Gen } r)}} \longrightarrow \text{Bew}_\kappa[Sb(r_{Z(x)}^{17})]$
(II)	$\overline{\overline{xB_\kappa(17 \text{ Gen } r)}} \longrightarrow \text{Bew}_\kappa[\text{Neg}(Sb(r_{Z(x)}^{17}))]$

---

(15) is false

(16) is false

FIGURE 3. (15) and (16) are false